Chapter 3

Log geometry

3.1 A brief review of toric geometry

Recall the map $r : T_\Sigma \to M$ defined by $r(t_\rho) = m_\rho$, where m_ρ is a primitive generator of ρ. The transpose map ${}^t r : N \to \check{T}_\Sigma$ is given by

$$n \mapsto \sum_{\rho \in \Sigma^{[1]}} \langle n, m_\rho \rangle t_\rho^*$$

Then $\mathrm{Cl}(X_\Sigma) \simeq \operatorname{coker} {}^t r$. Here, an element $\psi : T_\Sigma \to \mathbb{Z}$ of T_Σ corresponds to the Weil divisor $\sum_{\rho \in \Sigma^{[1]}} \psi(t_\rho) D_\rho$, and ${}^t r(n)$ is the divisor of zeroes and poles of z^n. If r is in fact surjective, which is the case, for example, if X_Σ is non-singular and proper over $\operatorname{Spec} k$, we get an exact sequence

$$0 \to K_\Sigma \to T_\Sigma \to M \to 0 \tag{3.1}$$

Then the dual exact sequence is

$$0 \to N \to \check{T}_\Sigma \to \mathrm{Cl}(X_\Sigma) \to 0, \tag{3.2}$$

A divisor induced by $\psi \in \check{T}_\Sigma$ is Cartier if ψ is induced by a PL function $\phi : |\Sigma| \to \mathbb{R}$ with $\phi(m_\rho) = \psi(t_\rho)$.

Given a Cartier divisor D defined by a PL function ϕ, the divisor D is very ample if and only if ϕ is strictly convex.

Chapter 4

Mikhalkin's curve counting formula

4.1 The statement and outline of the proof

4.1.1 Notation

1. $M = \mathbb{Z}^n$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, Z)$
2. Σ: complete fan in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$
3. $\rho \in \Sigma^{[1]} \longrightarrow T_\Sigma =$ free group on $\Sigma^{[1]} = \mathbb{Z}\langle t_\rho : \rho \in \Sigma^{[1]}\rangle$
4. $r : T_\Sigma \to M$, $\Delta = \sum_{\rho \in \Sigma^{[1]}} d_\rho t_\rho \in \mathrm{Ker}\, r$
5. $d_\rho =$ number of edges in direction ρ in a tropical curve
6. $h : \Gamma \to M_{\mathbb{R}}$: simple tropical curve, $\dim M_{\mathbb{R}} = 2$
7. given fan Σ, degree Δ, h passing through $|\Delta| - 1$ points $\in M_{\mathbb{R}}$

$$N^{0,\mathrm{trop}}_{\Delta,\Sigma} = \sum_{h \in \mathcal{M}_{0,|\Delta|-1}(\Sigma,\Delta)} \mathrm{Mult}(h)$$

8. $V \in \Gamma^{[0]}$ adjacent edges E_1, E_2, E_3

$$\begin{aligned} \mathrm{Mult}_V(h) &= w_\Gamma(E_1) w_\Gamma(E_2) |m_{(V,E_1)} \wedge m_{(V,E_2)}| \\ &= w_\Gamma(E_2) w_\Gamma(E_3) |m_{(V,E_2)} \wedge m_{(V,E_3)}| \\ &= w_\Gamma(E_3) w_\Gamma(E_1) |m_{(V,E_3)} \wedge m_{(V,E_1)}| \end{aligned}$$

if none of E_1, E_2, E_3 are marked, and otherwise $\mathrm{Mult}_V(h) = 1$