

# Drawing a surface in $\mathbb{L}^3$

Young Wook Kim

Nov., 2004

This paper is to introduce a known method of drawing minimal surfaces in  $\mathbb{E}^3$  and show how to use it to find and draw a family of maximal surfaces in the Minkowski space  $\mathbb{L}^3$ .

Our question to begin with was "Is there a timelike maximal surface in  $\mathbb{L}^3$  similar to the Costa-Hoffman-Meeks surfaces?"

## 1 Maximal surfaces in $\mathbb{E}^3$

Minkowski space-time  $\mathbb{L}^3$  is  $\mathbb{R}^3 = \{(x, y, t)\}$  with pseudo-riemannian metric  $dx^2 + dy^2 - dt^2$ . Let  $M$  be a Riemann surface, and  $f, g : M \rightarrow \mathbb{C}$  be analytic functions. Then, the Weierstrass-type formula

$$Re \left\{ \int_{z_0}^z ((1 + g(w)^2)f(w), i(1 - g(w)^2)f(w), 2g(w)f(w)) dw \right\}$$

defines a space-like maximal immersion into  $\mathbb{L}^3$ .

The following table is a list of maximal surfaces in  $\mathbb{L}^3$ .

Surfaces	Riemann Surface $M$	$f dz$	$g$
catenoid	$\mathbb{S}^2 \setminus \{0, \infty\}$	$\frac{1}{z^2} dz$	$z$
helicoid	$\mathbb{S}^2 \setminus \{0, \infty\}$	$\frac{i}{z^2} dz$	$z$
Enneper's surface	$\mathbb{C}$	$dz$	$z$
Trinoid	$\mathbb{S}^2 \setminus \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$	$\frac{1}{(z^3 - 1)^2} dz$	$z^2$
Costa's surface	?	?	?

## 2 The process of finding a surface

## 3 Domain of parametrization

## 4 The Weierstrass data

Data of Costa-Hoffman-Meeks minimal surfaces in  $\mathbb{E}^3$  is

$$w^{k+1} = z^k(z^2 - 1), \quad \eta = \left(\frac{z}{w}\right)^k dz, \quad g = \frac{\rho}{w}.$$

Literal conversion of this data into the maximal surface does not give a closed surface. And we have to modify it hoping for a data that works. The data we found for our maximal surfaces in  $\mathbb{L}^3$  is:

$$w^{k+1} = z^k(z^2 - 1), \quad \eta = \frac{1}{z} \left(\frac{z}{w}\right)^k dz, \quad g = \rho \frac{z}{w}$$

## 5 The period problem

For the periods of other components we have

**Lemma 1.** *The period around  $\gamma$  for the  $xy$ -components are 0 iff*

$$\sigma = \sqrt{\frac{1}{2} \frac{A}{B}},$$

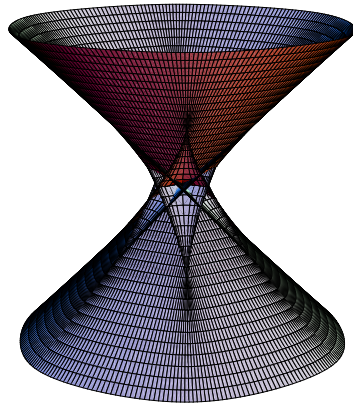
where

$$A = \int_0^1 \frac{dt}{\sqrt[2k+1]{t^k(1-t^2)}}, \quad B = \int_0^1 \frac{\sqrt[2k+1]{t^k(1-t^2)} dt}{1-t^2}.$$

## 6 Symmetries of the surface

## 7 The Surface

We render the surface graphics using a 3D plotting function in Mathematica and here is the surface:



## 8 Geometric properties of the surface

The metric of the surface is

$$ds^2 = (1 - |g|^2)^2 |\eta|^2,$$

and the singularities occur at the points where  $|g| = 1$ .

In polar coordinates for  $\alpha = r e^{i\theta}$ , the singularity set is given by the equation

$$r^2 + r^{-2} = \sigma^{-2(k+1)} + 2 \cos 2\theta.$$

## 9 Final remarks

This gives a new surface which is in close relation with the Costa-Hoffman-Meeks minimal surface.

## References

- [1] Hoffman, D. and W. H. Meeks, III, Embedded minimal surfaces of finite topology, *Ann. of Math.* **131** (1990), 1–34.