Drawing a surface in $\mathbb{L}^3$

Young Wook Kim

Nov., 2004

This paper is to introduce a known method of drawing minimal surfaces in $\mathbb{E}^3$ and show how to use it to find and draw a family of maximal surfaces in the Minkowski space $\mathbb{L}^3$.

Our question to begin with was "Is there a spacelike maximal surface in $\mathbb{L}^3$ similar to the Costa-Hoffman-Meeks surfaces?" [1]

1 Maximal surfaces in $\mathbb{E}^3$

Minkowski space-time $\mathbb{L}^3$ is $\mathbb{R}^3 = \{(x, y, t)\}$ with pseudo-riemannian metric $dx^2 + dy^2 - dt^2$. Let M be a Riemann surface, and $f, g : M \to \mathbb{C}$ be analytic functions. Then, the Weierstrass-type formula

$$Re\left\{\int_{z_0}^{z} \big((1 + g(w)^2)f(w), \boldsymbol{i}\,(1 - g(w)^2)f(w), 2\,g(w)f(w)\big)\,dw\right\}$$

defines a space-like maximal immersion into $\mathbb{L}^3$.

The table in the following page is a list of maximal surfaces in $\mathbb{L}^{\mathbf{3}}$.

2 The process of finding a surface

3 Domain of parametrization

4 The Weierstrass data

Data of Costa-Hoffman-Meeks minimal surfaces in $\mathbb{E}^3$ is

$$w^{k+1} = z^k(z^2 - 1), \quad \eta = \left(\frac{z}{w}\right)^k dz, \quad g = \frac{\rho}{w}.$$

Surfaces	**Riemann Surface M**	$\boldsymbol{f\,dz}$	$\boldsymbol{g}$
catenoid	$\mathbb{S}^2 \setminus \{0,\infty\}$	$\frac{1}{z^2}\,dz$	z
helicoid	$\mathbb{S}^2 \setminus \{0,\infty\}$	$\frac{i}{z^2}\,dz$	z
Enneper's surface	$\mathbb{C}$	dz	z
Trinoid	$\mathbb{S}^2 \setminus \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$	$\frac{1}{(z^3-1)^2}\,dz$	z^2
Costa's surface	?	?	?

Table 1: A table

Literal conversion of this data into the maximal surface does not give a closed surface. And we have to modify it hoping for a data that works. The data we found for our maximal surfaces in $\mathbb{L}^3$ is:

$$w^{k+1} = z^k(z^2-1), \quad \eta = \frac{1}{z}\left(\frac{z}{w}\right)^k dz, \quad g = \rho\,\frac{z}{w}$$

5 The period problem

For the periods of other components we have

Lemma 1. *The period around γ for the xy-components are 0 iff*

$$\sigma = \sqrt{\frac{1}{2}\frac{A}{B}},$$

where

$$A = \int_0^1 \frac{dt}{\sqrt[k+1]{t^k(1-t^2)}}, \quad B = \int_0^1 \frac{\sqrt[k+1]{t^k(1-t^2)}dt}{1-t^2}.$$

6 Symmetries of the surface

7 The Surface

We render the surface graphics using a 3D plotting function in Mathematica and here is the surface:

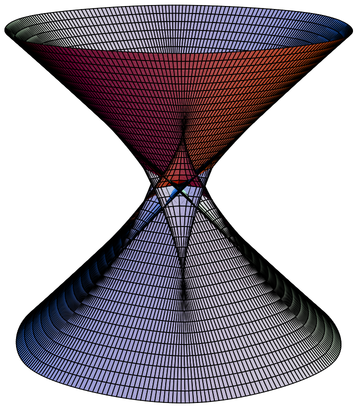

Figure 1: A figure

8 Geometric properties of the surface

The metric of the surface is

$$ds^2 = (1 - |g|^2)^2 |\eta|^2,$$

and the singularities occur at the points where $|g| = 1$.

In polar coordinates for $\alpha = r\, e^{i\theta}$, the singularity set is given by the equation

$$r^2 + r^{-2} = \sigma^{-2(k+1)} + 2\cos 2\theta.$$

9 Final remarks

This gives a new surface which is in close relation with the Costa-Hoffman-Meeks minimal surface.

References

[1] Hoffman, D. and W. H. Meeks, III, Embedded minimal surfaces of finite topology, *Ann. of Math.* **131** (1990), 1–34.